MTF | Mat 2 | Čas 1 ①

Diferencijalne j-ne I-og reda

Def J-na oblika $F(x, y, y', \ldots, y^{(n)}) = 0$, $y = y(x)$ nazva se difer. jednačinom

Dif. j-ne koje razdvajaju prom.

$$y' = f(x)\cdot g(y), \quad f, g \in C, \quad y = y(x).$$

$$\frac{dy}{dx} = f(x)\cdot g(y), \quad g(y) \neq 0$$

$$\frac{dy}{g(y)} = f(x)dx \Rightarrow \int \frac{dy}{g(y)} = \int f(x)dx$$

$$\underline{G(y) = F(x) + c}$$

$x^3y^2y' = 1 - x^2$

1) $y' = 2xy$ $\quad y' = 5x^2y^{-1}$

$$\frac{dy}{dx} = 2xy$$

$$\frac{dy}{y} = 2x\,dx$$

$$\int \frac{dy}{y} = \int 2x\,dx$$

$$\ln|y| = 2\frac{x^2}{2} + c$$

$$\ln|y| = x^2 + c$$

$$|y| = e^{x^2+c}$$

$$y = \pm e^{x^2+c}, \quad \pm e^c = A$$

$$\boxed{y = Ae^{x^2}}$$

2) $y' = 1 + x + y + xy$ $\quad 2 + x + 2y + xy$

$$\frac{dy}{dx} = 1 + x + y(1+x)$$

$$\frac{dy}{dx} = (1+x)(1+y)$$

$$\frac{dy}{1+y} = (1+x)dx$$

$$\int \frac{dy}{1+y} = \int (1+x)dx$$

$$\ln|1+y| = x + \frac{x^2}{2} + c$$

$$|1+y| = e^{x + \frac{x^2}{2} + c}$$

$$\underline{y = \pm e^{x + \frac{x^2}{2} + c} - 1}$$

3) $xyy' = 1 - x^2$

$$xy\cdot\frac{dy}{dx} = 1 - x^2$$

$$y\,dy = \frac{1-x^2}{x}\cdot dx$$

$$\int y\,dy = \int \frac{1-x^2}{x}\,dx$$

$$\int y\,dy = \int \frac{dx}{x} - \int x\,dx$$

$$\frac{y^2}{2} = \ln|x| - \frac{x^2}{2} + c$$

$$y^2 = 2\left(\ln|x| - \frac{x^2}{2} + c\right)$$

$$\underline{y = \pm\sqrt{2\left(\ln|x| + \frac{x^2}{2} + c\right)}}$$

4) $y' \operatorname{tg} x = y$

$$\frac{dy}{dx} = \frac{y}{\operatorname{tg}x}$$

$$\frac{dy}{y} = \frac{dx}{\operatorname{tg}x}$$

$$\int \frac{dy}{y} = \int \frac{dx}{\operatorname{tg}x}$$

$$\ln|y| = \int \frac{dx \cdot \cos x}{\sin x}$$

$\sin x = t$

$\cos x\, dx = dt$

$$\ln|y| = \int \frac{dt}{t}$$

$$\ln|y| = \ln|t| + c$$

$$\ln|y| = \ln|\sin x| + c$$

$$|y| = e^{\ln|\sin x|} \cdot A$$

$$|y| = |\sin x| \cdot A$$

5) $(1+x)y' + 1 + y = 0$

$$(1+x)\frac{dy}{dx} = -1-y$$

$$\frac{dy}{-1-y} = \frac{dx}{1+x}$$

$$-\int \frac{dy}{1+y} = \int \frac{dx}{1+x}$$

$$\ln|1+y| = -\ln|1+x| + c$$

$$|1+y| = e^{-\ln|1+x|+c}$$

$$y = \pm e^{-\ln|1+x|+c} - 1$$

6) $xy' - y = y^3$

$$x \cdot \frac{dy}{dx} = y^3 + y$$

$$\frac{x}{dx} = \frac{y^3+y}{dy}$$

$$\frac{dx}{x} = \frac{dy}{y^3+y}$$

$$\int \frac{dx}{x} = \int \frac{dy}{y(y^2+1)} \quad (1)$$

$$I_1 = \int \frac{dy}{y(y^2+1)}$$

$$\frac{1}{y(y^2+1)} = \frac{A}{y} + \frac{By+C}{y^2+1}$$

$$1 = Ay^2 + A + By^2 + Cy$$

$A + B = 0 \Rightarrow B = -1$

$A = 1$

$C = 0$

$$I_1 = \int \frac{dy}{y} + \int \frac{-y}{y^2+1}\, dy =$$

$y^2 + 1 = t$

$2y\, dy = dt$

$y\, dy = \frac{dt}{2}$

$$= \ln|y| - \frac{1}{2}\int \frac{dt}{t} = \ln|y| - \frac{1}{2}\ln|y^2+1|$$

$(1) \Leftrightarrow \ln|x| = \ln|y| - \frac{1}{2}\ln|y^2+1| + c$

$$\ln|x| = \ln\frac{y}{\sqrt{y^2+1}} + c$$

$$\frac{y}{\sqrt{y^2+1}} = |x| \cdot e^c$$

7) $y' = 10^{x+y}$

$$\frac{dy}{dx} = 10^x \cdot 10^y$$

$$\frac{dy}{10^y} = 10^x dx$$

$$\int \frac{dy}{10^y} = \int 10^x dx$$

$$\int 10^{-y} dy = \int 10^x dx$$

$$-\frac{10^{-y}}{\ln 10} = \frac{10^x}{\ln 10} + C$$

8) $y' = \frac{1+y^2}{(1+x^2)xy}$

$$\frac{dy}{dx} = \frac{1+y^2}{y} \cdot \frac{1}{(1+x^2)x}$$

$$\frac{y\,dy}{1+y^2} = \frac{1}{(1+x^2)x} dx$$

$$\int \frac{y\,dy}{1+y^2} = \int \frac{dx}{x(1+x^2)}$$

$$\frac{1}{2}\ln|1+y^2| = \ln|x| - \frac{1}{2}\ln|x^2+1| + C_1$$

$$\sqrt{1+y^2} = \frac{x}{\sqrt{x^2+1}} e^c$$

9) a) $\sqrt{1-x^2}\,dy - x\sqrt{1-y^2}\,dx = 0$ (ngā sv.)

b) $\frac{dy}{\sqrt{x}} = \frac{3dx}{\sqrt{y}}$ (ngā sv.)

c) $yy' = xe^{x^2+y^2}$ (nga sv.)

d) $(y^2+xy^2)y' + x^2 - yx^2 = 0$ (nga sv.)

10) $(1+e^x)yy' = e^y$, $y(0)=0$

R:

$$(1+e^x)\,y \cdot \frac{dy}{dx} = e^y$$

$$\int \frac{y}{e^y} dy = \int \frac{dx}{1+e^x}$$

$$I_1 = \int \frac{y\,dy}{e^y}$$

$y = u$ $\quad \frac{dy}{e^y} = dv$

$dy = du$ $\quad v = \int \frac{dy}{e^y} = \underline{\underline{-e^{-y}}}$

$$I_1 = -ye^{-y} + \int e^{-y} dy =$$

$$= -ye^{-y} - e^{-y} = \underline{e^{-y}(-y-1)}$$

$$I_2 = \int \frac{dx}{1+e^x}, \quad e^x = t$$

$x = \ln t$

$dx = \frac{dt}{t}$.

$$I_2 = \int \frac{dt}{t(1+t)}$$

$$\frac{1}{t(1+t)} = \frac{A}{t} + \frac{B}{1+t} \quad \ldots$$

$$I_2 = \ln|t| - \ln|1+t| + c =$$

$$= \underline{\ln e^x - \ln|1+e^x| + c}$$

$I_1 = I_2 \Rightarrow -e^{-y}(y+1) = \ln e^x - \ln(1+e^x) + C$

$y(0)=0 \qquad -e^0(0+1) = \underbrace{\ln e^0}_{=0} - \ln(1+e^0) + C$

$$-1 = -\ln 2 + C$$

$$\boxed{C = \ln 2 - 1}$$

$$\underline{-e^{-y}(y+1) = \ln e^x - \ln(1+e^x) + \ln 2 - 1}.$$

(11) nograd. a) $y' + y^2 = 1,\ y(0)=0$, b) $e^x y' = xy^2,\ y(1)=1$.

Homogene dif. j-ne

(1) $y' = g\left(\frac{y}{x}\right)$ smjene $\frac{y}{x} = u,\ u = u(x)$

$y = u\cdot x \Rightarrow y' = u'\cdot x + u$

(1) $\Leftrightarrow u'x + u = g(u)$

$$\frac{du}{dx}x = g(u) - u$$

$$\int \frac{du}{g(u)-u} = \int \frac{dx}{x}$$

1) $y' = \frac{x+y}{x}$

$y' = 1 + \frac{y}{x},\ \frac{y}{x} = u$

$u'x + u = 1 + u$

$u'x = 1$

$\frac{x\,du}{dx} = 1$

$\int du = \int \frac{dx}{x}$

$u = \ln|x| + C$

$\frac{y}{x} = \ln|x| + C$

$y = x(\ln|x| + C)$

2) $x^2 y' = xy + x^2 + y^2 \ /: x^2$

$y' = \frac{y}{x} + 1 + \left(\frac{y}{x}\right)^2,\quad u = \frac{y}{x},\ y = u\cdot x$

$u'x + \cancel{u} = \cancel{u} + 1 + u^2$

$u'x = 1 + u^2$

$\frac{du}{dx}\cdot x = 1 + u^2$

$\int \frac{dx}{x} = \int \frac{du}{1+u^2}$

$\ln|x| = \operatorname{arctg} u$

$$\underline{\operatorname{arctg}\frac{y}{x} = \ln|x| + C}.$$

3) $(x^2+y^2)dx - 2xy\,dy = 0 \quad /:x^2$

$\left(1+\left(\frac{y}{x}\right)^2\right)dx - 2\frac{y}{x}dy = 0$

$\left(\frac{y}{x}\right) = u \Rightarrow y = xu$

$(1+u^2)dx - 2u\,dy = 0$

$1+u^2 = 2u\left(\frac{dy}{dx}\right) = y'$

$1+u^2 = 2u\cdot(u'x+u)$

$1+u^2 = 2uu'x + 2u^2$

$1-u^2 = 2u\frac{du}{dx}\cdot x$

$\int\frac{dx}{x} = \int\frac{2u\,du}{1-u^2} \quad \ulcorner 1-u^2 = t \lrcorner$

$\ln|x| + c = -\ln|1-u^2|$

$\underline{\ln|x| = -\ln\left|1-\left(\frac{y}{x}\right)^2\right| + c}$

4) $xy' = y\left(1+\ln\frac{y}{x}\right)$

$y' = \frac{y}{x}\left(1+\ln\frac{y}{x}\right)$

$\frac{y}{x} = u$

$u'x + u = u(1+\ln u)$

$x\frac{du}{dx} + \cancel{u} = \cancel{u} + u\ln u$

$\frac{x\,du}{dx} = u\ln u$

$x\,du = u\ln u\,dx$

$\int\frac{du}{u\ln u} = \int\frac{dx}{x}$

$\ln u = m$
$\frac{du}{u} = dm$
$\Rightarrow I_1 = \int\frac{dm}{m} =$

$= \underline{\ln|\ln(u)|}$

$\ln|\ln(u)| = \ln|x| + c$

$\ln\left|\ln\frac{y}{x}\right| = \ln|x|\cdot e^c \Rightarrow \underline{\ln\frac{y}{x} = |x|\cdot e^c}$

5) za vj.:

a) $3y^2 - 2xyy' + x^2 = 0$ (Jelino) $/:x^2$

$3\frac{y^2}{x^2} - 2y'\frac{y}{x} + 1 = 0$

$\frac{y}{x} = u$

$y = ux$

$3u^2 - 2(u'x+u)u + 1 = 0$

$u^2 - 2\frac{du}{dx}xu + 1 = 0$

$u^2+1 = 2\frac{xu\,du}{dx} \quad /\cdot u\,du$

$\int 2\frac{u\,du}{u^2+1} = \int\frac{dx}{x}$

b) $(y + \sqrt{x^2+y^2})dx - x\,dy = 0$ (moja sv). $/:x$

$\left(\frac{y}{x} + \sqrt{1+\left(\frac{y}{x}\right)^2}\right)dx - 1\,dy = 0 \quad /:dx$

c) $xy^2dy = (x^3+y^3)dx$ (nga sv.)

Linearne dif. j-ne

Opšti oblik lin. dif. j-ne prvog reda: $y' + yP(x) = Q(x)$

$$\frac{y'}{y} + P(x) - \frac{Q(x)}{y} = 0$$

$$\frac{y'}{y} = P(x) = \frac{z'}{z}, \; z = z(x), \quad \frac{z'}{z} - \frac{Q(x)}{y} = 0 \quad (*)$$

Integraljenjem dobijamo:

$$\ln y - \ln z = -\int P(x)\,dx \Rightarrow \ln\frac{y}{z} = -\int P(x)\,dx \Rightarrow$$

$$y = ze^{-\int P(x)\,dx}$$

u $(*) \Rightarrow \quad z = \int Q(x)\,e^{\int P(x)\,dx}\,dx + C$

Konačno $\underline{y = e^{-\int P(x)\,dx}\left(\int Q(x)\,e^{\int P(x)\,dx}\,dx + C\right)}$.

1) $(1-x^2)y' + 2xy - 4x = 0 \quad /:(1-x^2)$

$$y' + \frac{2xy}{1-x^2} - \frac{4x}{1-x^2} = 0$$

$$y' + \frac{2x}{1-x^2}y = \frac{4x}{1-x^2}$$

$$P(x) = \frac{2x}{1-x^2}, \quad Q(x) = \frac{4x}{1-x^2}$$

$$y = e^{-\int P(x)\,dx}\left(\int Q(x)\,e^{\int P(x)\,dx}\,dx + C\right)$$

$$\int \frac{2x}{1-x^2}\,dx = \begin{bmatrix} 1-x^2 = t \\ -2x\,dx = dt \end{bmatrix} = \int -\frac{dt}{t} = -\ln|1-x^2|$$

(5)
(4)

$$\int \frac{4x}{1-x^2} e^{-\ln|1-x^2|} dx = \int \frac{4x}{1-x^2} \cdot \frac{1}{1-x^2} dx$$

$(1-x^2) = t$

$-2x\,dx = dt$

$x\,dx = -\frac{dt}{2}$

$$\Rightarrow \int \frac{4 \cdot -\frac{dt}{2}}{t^2} = -2\int \frac{dt}{t^2} =$$

t^{-2}

$$= -2 \cdot \frac{t^{-1}}{-1} + c = \frac{2}{1-x^2} + c$$

$$y = (1-x^2)\left(\frac{2}{1-x^2} + c\right) = 2 + c(1-x^2)$$

2) $xy' - y + \ln x = 0 \quad / : x \quad / \; y(1) = 2$

$y' - \frac{y}{x} + \frac{\ln x}{x} = 0$

$y' - \frac{1}{x} \cdot y = -\frac{\ln x}{x}$

$P(x) = -\frac{1}{x}, \quad Q(x) = -\frac{\ln x}{x}$

$$y = e^{-\int P(x)dx}\left(\int Q(x) e^{\int P(x)dx} + c\right)$$

$\int -\frac{1}{x} dx = -\ln|x|$ ✓

$$\int -\frac{\ln x}{x} \cdot e^{-\ln|x|} = \int -\frac{\ln x}{x} \frac{1}{x} dx = -\int \frac{\ln x}{x^2} dx$$

$$I = \int \frac{\ln x}{x^2} dx = \quad \ln x = u \quad \frac{dx}{x^2} = dv$$

$$\frac{dx}{x} = du \quad \Rightarrow v = \int \frac{dx}{x^2} = -\frac{1}{x}$$

$$\Rightarrow I = -\frac{1}{x}\ln x + \int \frac{1}{x} \frac{dx}{x} = -\frac{1}{x}\ln x - \frac{1}{x} + c$$

$$y = e^{\ln|x|}\left(+\frac{1}{x}\ln x + \frac{1}{x} + c\right) = \ln x + 1 + cx$$

$2 = 0 + 1 \cdot c \cdot 1 \Rightarrow c + 1 = 2$

$\boxed{c = 1}$

3) $y' + y\,tg x = \frac{1}{\cos x}$

$P(x) = tg x, \quad Q(x) = \frac{1}{\cos x},$

$y = e^{-\int P(x)dx}\left(\int Q(x) e^{\int P(x)dx} dx + C\right)$

$\int tg x\, dx = \int \frac{\sin x}{\cos x} dx = \left\lceil \begin{matrix}\cos x = t \\ -\sin x dx = dt\end{matrix} \right\rfloor = \int -\frac{dt}{t} = \underline{-\ln|t|} =$
$= -\ln|\cos x|$

$\int \frac{1}{\cos x} e^{-\ln(\cos x)} dx = \int \frac{1}{\cos^2 x} dx = tg x + c$

$y = \cos x\,(tg x + c) = \underline{\sin x + c\cos x}$

④ $y' + y\,tg x - \sin 2x = 0$ $\qquad P(x) = tg x, \quad Q(x) = \sin 2x$ (Jelmod

$y = e^{-\int P(x)dx}\left(\int Q(x) e^{\int P(x)dx} + c\right)$

5) $(\sin^2 y + x\,ctg y)\, y' = 1$

$\int tg x dx = -\ln|\cos x|$

Posmotrajmo $\quad x = x(y), \quad y' = \frac{1}{x'}$
$1 = x' \cdot y' \Rightarrow y' = \frac{1}{x'}$

$\int \sin 2x\, e^{\ln\frac{1}{\cos x}} dx =$

$\Rightarrow (\sin^2 y + x\,ctg y)\cdot \frac{1}{x'} = 1 \rightarrow$

$= \int 2\sin x \cancel{\cos x} \frac{1}{\cancel{\cos x}} dx =$

$x' - x\,ctg y = \sin^2 y, \quad \begin{cases} P(y) = -ctg y \\ Q(y) = \sin^2 y \end{cases}$

$= \underline{2(-\cos x)}$

$\underline{x = \sin y(-\cos y + C)},$

$y = \cos x(2 - \cos x) + C$

$y' - y\,ctg x = \sin^2 x \qquad \begin{matrix} P(x) = -ctg x \\ Q(x) = \sin^2 x \end{matrix}$

$\lceil y' - y\,ctg x = \sin^2 x \rfloor$

Beenulijeva j-na

Opšti oblik $y' + P(x) y = Q(x) \cdot y^{\alpha}$

$\alpha = 0$ i $\alpha = 1 \rightarrow$ lin. nehom. dif. j-na i lin. homog. dif. j-na.

$\alpha \neq 0 \wedge \alpha \neq 1 \rightarrow z = y^{1-\alpha} \Rightarrow$

$$\frac{1}{1-\alpha} \frac{dz}{dx} + P(x) z = Q(x).$$

$y = uv$ ili varij. konstanti.

1) $xy' + y - y^2 \ln x = 0$

$xy' + y = y^2 \ln x \; / : x \quad P(x) =$

$y' + \frac{1}{x} y = y^2 \cdot \frac{\ln x}{x}, \quad P(x) = \frac{1}{x}, \quad Q(x) = \frac{\ln x}{x}, \quad \alpha = 2$

Smjena: $z = y^{1-\alpha} \Rightarrow z = y^{1-2} = \frac{1}{y} \Rightarrow y = \frac{1}{z}$

$y' = -\frac{1}{z^2} \cdot z'$

$-\frac{z'}{z^2} + \frac{1}{x} \cdot \frac{1}{z} = \frac{1}{z^2} \frac{\ln x}{x} \; / \cdot (-z^2)$

$z' - z \cdot \frac{1}{x} = -\frac{\ln x}{x}$

$z = e^{-\int -\frac{dx}{x}} \left(\int -\frac{\ln x}{x} \cdot e^{\int -\frac{dx}{x}} dx + C \right) \cdot \ldots$

$z = Cx + 1 + \ln|x| \quad ; \quad \int \frac{dx}{x} = \ln|x|$

$z = y^{-1} \Rightarrow y = \frac{1}{Cx + 1 + \ln|x|} \quad \Big| \quad -\int \frac{\ln x}{x} \cdot e^{\ln x^{-1}} dx =$

$= -\int \frac{\ln x}{x^2} dx = \left\lceil \begin{matrix} \ln x = u \\ \frac{dx}{x^2} = dv \end{matrix} \right\rfloor$

2) $xy' - 4y = x\sqrt{y}$ $\quad / : x\sqrt{y}$

$\frac{y'}{\sqrt{y}} - \frac{4}{x}\sqrt{y} = x$

$\sqrt{y} = z \Rightarrow z' = \frac{1}{2\sqrt{y}}\, y'$

$\frac{2\cancel{\sqrt{y}}\cdot z'}{\cancel{\sqrt{y}}} - \frac{4}{x} z = x$

$2z' - \frac{4}{x} z = x$

$z' - \frac{2}{x} z = \frac{x}{2}$, $P(x) = -\frac{2}{x}$

$Q(x) = \frac{x}{2}$

$/ : x$ $\quad \cancel{x}\, y' - \frac{4}{x}\cdot y = x\, y^{1/2}$

$z = y^{1/2} \Rightarrow$

$y = z^2 \Rightarrow y' = 2z\cdot z'$

$2zz' - \frac{4}{x}\cdot z^2 = x\cdot z \quad / : 2z$

$z' - \frac{2}{x} z = \frac{x}{2}$

$P(x) = -\frac{2}{x}$, $Q(x) = \frac{x}{2}$

$\int -\frac{2}{x}dx = -2\ln|x|$,

$\int \frac{x}{2} e^{\ln x^{-1}} = \int \frac{1}{2x}dx = \frac{1}{2}\ln|x| + C$

$\int -\frac{2}{x}dx = -2\ln|x|$

$\int Q(x)\, e^{\int P(x)dx} dx = -\int \frac{x}{2}\frac{1}{x^2}dx = -\frac{1}{2}\ln|x|$

$z = e^{-\int P(x)dx}\left(\int Q(x)\, e^{\int P(x)dx}dx + C\right) =$

$= e^{\ln x^2}\left(-\frac{1}{2}\ln|x| + C\right) = \frac{1}{2}x^2\ln|x| + x^2 C$

$y = z^2 = \frac{1}{4}x^4\ln^2|x| + x^4C^2 + x^4\ln|x|\,C$

3) $y' + y = xy^3 \quad / : y^3$, $P(x) = 1$, $Q(x) = x$, $\alpha = 3$

$z = y^{1-3} = y^{-2}$ $\to$ $z = \frac{1}{y^2}$, $z' = -\frac{2}{y^3}\cdot y'$

$\frac{y'}{y^3} + \frac{1}{y^2} = x$

$-\frac{z'}{2} + z = x$

$z' - 2z = -2x$, $P(x) = -2$, $Q(x) = -2x$.

1) $y' + y\,tg\,x = \sin 2x$

$P(x) = tg\,x, \quad Q(x) = \sin 2x$

$$y = e^{-\int p(x)dx}\left(\int Q(x)\, e^{\int p(x)dx}\, dx + C\right)$$

$$\int tg\,x\, dx = \int \frac{\sin x}{\cos x}\, dx = \left[\begin{array}{l} \cos x = t \\ -\sin x\, dx = dt \end{array}\right] = \int \frac{-dt}{t} = -\ln|t| =$$

$$= -\ln|\cos x|$$

$$\int \sin 2x\, e^{-\ln|\cos x|}\, dx = \int \sin 2x\, \frac{1}{\cos x}\, dx =$$

$$= \int 2\sin x \cancel{\cos x}\, \frac{1}{\cancel{\cos x}}\, dx = 2\int \sin x\, dx = \underline{\underline{-2\cos x}}$$

$$y = e^{\ln|\cos x|}\left(-2\cos x + C\right) = \underline{\cos x\left(-2\cos x + C\right)}$$

a) $(y^2+xy^2)y'+x^2-yx^2=0$

$(y^2+xy^2)\frac{dy}{dx}=yx^2-x^2$

$y^2(1+x)\frac{dy}{dx}=x^2(y-1)$

$\int\frac{y^2}{y-1}dy=\int\frac{x^2}{1+x}dx$

$$\begin{array}{l} x^2:(x+1)=x-1+\frac{1}{x+1} \\ x^2+x \\ \hline -x \\ -x-1 \\ \hline 1 \end{array}$$

$\int\frac{x^2+1-1}{1+x}dx=$

$=\int\frac{(x-1)(x+1)}{x+1}dx+$

$+\int\frac{dx}{1+x}$

$=\frac{x^2}{2}-x+\ln$

$\int x\,dx-\int dx+\int\frac{dx}{x+1}=$

$=\frac{x^2}{2}-x+\ln|x+1|$

$$\begin{array}{l} y^2:(y-1)=y+1 \\ y^2-y \\ \hline y \\ y-1 \\ \hline 1 \end{array}$$

$\Rightarrow\frac{y^2}{y-1}=y+1+\frac{1}{y-1}$

$\int y\,dy+\int dy+\int\frac{dy}{y-1}=$

$=\frac{y^2}{2}+y+\ln|y-1|$

Homogene

1) $3y^2-2xyy'+x^2=0\ /:x^2$

$\frac{3y^2}{x^2}-2\frac{y}{x}y'+1=0$

$\frac{y}{x}=u\Rightarrow y=ux$

$y'=u'x+u$

$3u^2-2u(u'x+u)+1=0$

$3u^2-2uu'x-2u^2+1=0$

$u^2=2uu'x-1\ \cancel{/:u}$

$\cancel{u=2u'x}$

$u^2+1=2uu'x\ /:u$

$\frac{u^2+1}{u}=2u'x$

$u'=\frac{du}{dx}$

$\frac{u^2+1}{u}=2\frac{du}{dx}x$

$\int\frac{2u\,du}{u^2+1}=\int\frac{dx}{x}$

$u^2+1=t$

$2u\,du=dt$

$\int\frac{dt}{t}=\ln|x|+c\Rightarrow\ln|u^2+1|=\ln|x|+c$

1) $\sqrt{1-x^2}\,dy - x\sqrt{1-y^2}\,dx = 0$

$\sqrt{1-x^2}\,dy = x\sqrt{1-y^2}\,dx$

$\frac{\sqrt{1-x^2}}{x}\,dy = \sqrt{1-y^2}\,dx$

$\frac{dy}{\sqrt{1-y^2}} = \frac{x\,dx}{\sqrt{1-x^2}}$

$\int \frac{dy}{\sqrt{1-y^2}} = \int \frac{x\,dx}{\sqrt{1-x^2}}$

$\arcsin y = \underbrace{\int \frac{x\,dx}{\sqrt{1-x^2}}}_{I}$

$1-x^2 = t$

$-2x\,dx = dt \Rightarrow$

$x\,dx = -\frac{dt}{2}$

$I = -\int \frac{dt}{2\sqrt{t}} = \frac{-1}{2}\int \frac{dt}{\sqrt{t}} =$

$= -\frac{1}{2}\int dt\, t^{-1/2} = -\frac{1}{2}\cdot\frac{t^{-1/2+1}}{-1/2+1} =$

$= -\frac{1}{\cancel{2}}\cdot\frac{\sqrt{t}}{\frac{1}{\cancel{2}}} = -\sqrt{t}$.

$\arcsin y = -\sqrt{1-x^2} + c$

2) $yy' = xe^{x^2+y^2}$

$y\frac{dy}{dx} = x\,e^{x^2}\cdot e^{y^2}$

$\int \frac{y}{e^{y^2}}\,dy = \int xe^{x^2}\,dx$

$y^2 = t \Rightarrow 2y\,dy = dt$

$\frac{1}{2}\int \frac{dt}{e^t} = \frac{1}{2}\int e^{-t}dt = -\frac{1}{2}e^{-y^2}$

$x^2 = m \Rightarrow 2x\,dx = dm$

$x\,dx = \frac{dm}{2}$

$\frac{1}{2}\int e^{m}\,dm = \frac{1}{2}e^{x^2}$

$-\frac{1}{2}e^{-y^2} = \frac{1}{2}e^{x^2} + c \quad / \cdot 2$

$e^{-y^2} = -e^{x^2} + C_1$

$-y^2 = \ln(-e^{x^2} + C_1)$

$y^2 = -\ln(-e^{x^2} + C_1)$

Tablica integrala

$$\int dx = x + C$$

$$\int x^a\,dx = \frac{x^{a+1}}{a+1} + C \qquad (a \neq -1)$$

$$\int \frac{dx}{x} = \ln|x| + C$$

$$\int e^x\,dx = e^x + C$$

$$\int a^x\,dx = \frac{a^x}{\ln a} + C \qquad (a > 0,\ a \neq 1)$$

$$\int \cos x\,dx = \sin x + C$$

$$\int \sin x\,dx = -\cos x + C$$

$$\int \frac{dx}{\cos^2 x} = \operatorname{tg} x + C$$

$$\int \frac{dx}{\sin^2 x} = -\operatorname{ctg} x + C$$

$$\int \operatorname{ch} x\,dx = \operatorname{sh} x + C$$

$$\int \operatorname{sh} x\,dx = \operatorname{ch} x + C$$

$$\int \frac{dx}{\operatorname{ch}^2 x} = \operatorname{th} x + C$$

$$\int \frac{dx}{\operatorname{sh}^2 x} = -\operatorname{cth} x + C$$

$$\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + C$$

$$\int \frac{dx}{1+x^2} = \operatorname{arctg} x + C$$

$$\int \frac{dx}{\sqrt{1+x^2}} = \operatorname{Arsh} x + C = \ln\left(x + \sqrt{1+x^2}\right) + C$$

$$\int \frac{dx}{\sqrt{x^2-1}} = \operatorname{Arch} x + C = \ln\left|x + \sqrt{x^2-1}\right| + C$$

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin\frac{x}{a} + C \qquad (a > 0)$$

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\operatorname{arctg}\frac{x}{a} + C \qquad (a > 0)$$

$$\int \frac{dx}{\sqrt{a^2+x^2}} = \ln\left(x + \sqrt{a^2+x^2}\right) + C \qquad (a > 0)$$

$$\int \frac{dx}{\sqrt{x^2-a^2}} = \ln\left|x + \sqrt{x^2-a^2}\right| + C \qquad (a > 0)$$

$$\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\left|\frac{x+a}{x-a}\right| + C \qquad (a > 0)$$